MATH 429 EXAM - 16/06/2025 (180 MINUTES)

No books, notes, or electronic devices (especially no phones) are permitted during this exam.

You must show your work to receive credit. **Justify everything**.

Do not unstaple the exam or reorder the pages. All problems must be solved within the space provided (right after the statement of the problem). If you need to use the extra pages at the end, then mention this clearly in the aforementioned space, so your grader knows that they have to also look at the end (they will not check the extra pages unless explicitly told to).

We will provide scratch paper (loose sheets) but do not write solutions on them. **Only the 16 pages of the booklet you're now reading will be graded**.

Please do not leave the room during the first and last 30 minutes of the exam.

Please keep your CAMIPRO face up on the table at all times.

Don't forget to write your name, SCIPER, and sign the exam.

There are 7 problems, worth 100 points in total.

NAME: ______________________________

SCIPER: ______________________________

SIGNATURE: ______________________________

PROBLEM 1

Consider the 3-dimensional Lie algebra $\mathfrak{g} = \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c$ with Lie bracket defined by

$$[a, b] = b, \qquad [a, c] = c, \qquad [b, c] = 0$$

Show that this Lie algebra is solvable, in two ways:

(a) by considering its derived series $\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \dots$ *(8 points)*

Solution. The subspace $[\mathfrak{g}, \mathfrak{g}]$ is generated by b and c, which in turn commute. Thus $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$.

(b) by verifying Cartan's criterion $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_x \mathrm{ad}_y) = 0$ for all $x \in \mathfrak{g}$, $y \in [\mathfrak{g}, \mathfrak{g}]$. *(7 points)*

Solution. We compute explicitly the adjoint action in the given basis. The relations yield

$$\mathrm{ad}_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathrm{ad}_b = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{ad}_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

As in the previous point, we observe that $[\mathfrak{g}, \mathfrak{g}]$ is generated by b and c. Then we check the aforementioned Cartan's criterion. By linearity, it is enough to perform the computation on basis of $\mathfrak{g}$ and of $[\mathfrak{g}, \mathfrak{g}]$. A straightforward computation gives

$$\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_a \mathrm{ad}_b) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_a \mathrm{ad}_c) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_b \mathrm{ad}_c) = 0.$$

PROBLEM 2

Consider the irreducible representation $\mathfrak{sl}_2 \curvearrowright L(n)$ with highest weight $n \in \mathbb{Z}_{\geq 0}$, and the Verma module (below, $\mathfrak{b}$ denotes the subalgebra of $\mathfrak{sl}_2$ spanned by E and H)

$$M(n) = U\mathfrak{sl}_2 \bigotimes_{U\mathfrak{b}} \mathbb{C}$$

generated by a single vector $v = 1 \otimes 1$ satisfying the relations $Ev = 0$ and $Hv = nv$.

(a) Construct an explicit basis of $M(n)$ as an infinite-dimensional vector space, and prove explicit formulas for the action of the operators $E, F, H \in \mathfrak{sl}_2$ in this basis. *(8 points)*

Solution. In the following we denote as $x \cdot w$ the action of $U\mathfrak{g}$ on $M(n)$, and we denote as xy the multiplication in $U\mathfrak{g}$.
Let $\mathfrak{n}^-$ be the subalgebra of $\mathfrak{sl}_2$ generated by F. Then isomorphism (183) in the Lecture Notes gives the explicit linear basis $(F^i \cdot v)_{i \in \mathbb{N}}$ for $M(n)$. The associative algebra $U\mathfrak{g}$ acts on $M(n)$ by left multiplication on the first factor, and the action the operator F is free. Thus

$$F \cdot (F^i \cdot v) = F^{i+1} \cdot v \quad .$$

Next, we use the PBW theorem to compute the action of the operators E and H. We have

$$H \cdot (F^i \cdot v) = HF^i \cdot v = FHF^{i-1} \cdot v - 2F^i \cdot v \quad .$$

Proceeding by induction and using the relation $H.v = nv$ we find

$$H \cdot (F^i \cdot v) = (n - 2i)F^i \cdot v \quad . \tag{1}$$

The action of E on the first basis element is by the definition $E \cdot v = 0$. We work out the E action on the other basis elements by means of the PBW theorem. Using the relations in the algebra $U\mathfrak{g}$ we have

$$E \cdot (F^i \cdot v) = FEF^{i-1} \cdot v + HF^{i-1} \cdot v \quad .$$

Then we proceed inductively, using the H action (1) and the relation $E \cdot v = 0$. Eventually we get

$$E \cdot (F^i \cdot v) = \sum_{j=0}^{i-1} (n - 2j)F^{i-1} \cdot v. \tag{2}$$

(b) Identify the kernel $K(n)$ of the surjection $M(n) \twoheadrightarrow L(n)$. *(5 points)*

(More specifically, you should prove that $K(n)$ is isomorphic to some representation of $\mathfrak{sl}_2$ which we have "named" in class, such as an irreducible representation or a Verma module)

Solution 1. The surjection $M(n) \twoheadrightarrow L(n)$ sends the subspace generated by the vectors $v_0 = v, v_1 = F \cdot v \dots, v_n = F^n \cdot v$ onto $L(n)$. Then, the kernel contains the basis elements of weight less then $-n$. Recalling (1), such elements are $v_{n+1}, v_{n+2}, \dots$. Thus

$$K(n) = \mathrm{Span}(v_{n+1}, v_{n+2}, \dots).$$

The subspace $K(n)$ inherits a submodule structure because it is the kernel of a module map. In particular, we see that $K(n)$ is an irreducible, infnite dimensional module cyclically generated by the highest weight vector v_{n+1}, thus

$$K(n) \cong M(-n-2). \tag{3}$$

Solution 2. The linear subspace subspace $K(n) = \mathrm{Span}(v_{n+1}, v_{n+2}, \dots)$ is invariant with respect to the $\mathfrak{sl}_2$ action worked out in point (a). In fact, it is clearly invariant for the action of F and H, and from (2) we get

$$E \cdot (F^{n+1} \cdot v) = \sum_{j=0}^{n} (n-2j) F^n v = 0,$$

where the last equality follows from the symmetry of the above sum. The $\mathfrak{sl}_2$ action from point (a) also tells us that the quotient module $M(n)/K(n)$ is isomorphic to $L(n)$. In fact, both $M(n)/K(n)$ and $L(n)$ have a linear basis where E acts by rising the weight, F acts by lowering the weight, and H acts diagonally. Furthermore, we can conclude that $K(n) \cong M(-n-2)$ as in solution 1.

(c) Recall that the character of a representation $\mathfrak{sl}_2 \curvearrowright V$ is the formal sum

$$\chi_V = \sum_{n \in \mathbb{C}} t^n \cdot \dim_{\mathbb{C}} \Big(\text{weight } n \text{ subspace of } V\Big)$$

(note that χ_V may be a power series if V is infinite dimensional). Calculate $\chi_{L(n)}$, $\chi_{M(n)}$ and $\chi_{K(n)}$, where $K(n)$ denotes the kernel from part (b). *(6 points)*

Solution 1. Every weight space of $M(n)$ is one dimensional. Thus

$$\chi_{M(n)} = t^n + t^{n-2} + \dots = \frac{t^n}{1-t^{-2}} \quad .$$

Isomorphism (3) tells us that

$$\chi_{K(n)} = \frac{t^{-n-2}}{1-t^{-2}} \quad .$$

Finally we compute the character of the finite dimensional module $L(n)$ as

$$\chi_{L(n)} = t^n + t^{n-2} + \dots + t^{-n} = \frac{t^n}{1-t^{-2}} - \frac{t^{-n-2}}{1-t^{-2}}.$$

Observe also that the short exact sequence

$$0 \to K(n) \to M(n) \to L(n) \to 0$$

implies that any two of the characters $\chi_{L(n)}, \chi_{M(n)}, \chi_{K(n)}$ determine the third one.

Solution 2. The exrcise can be solved also by using Weyl's character formula. In particular, the weight lattice of $\mathfrak{sl}_2$ is just $\mathbb{Z} \subset \mathbb{R}$, and the root lattice is $\{\pm 2\}$. The Weyl group is isomorphic to S_2, and it acts by changing sign. Then, we compute the character of $L(n)$ by substituting these information in Weyl's formula, expressed in terms of formal symbols $e^{weight\ n} = t^n$. We get

$$\chi_{L(n)} = \frac{1}{t - t^{-1}} \left(t^{n+1} - t^{-n-1} \right) = \frac{t^n - t^{-n-2}}{1 - t^{-2}} \quad .$$

PROBLEM 3

(a) We know that $SL_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ are complex Lie groups with Lie algebras $\mathfrak{sl}_n$, $\mathfrak{o}_n$, $\mathfrak{sp}_{2n}$, respectively. The latter are simple Lie algebras, so therefore

$$\mathfrak{g} = \mathfrak{sl}_{20} \oplus \mathfrak{o}_{25}$$

is a semisimple Lie algebra. Construct a complex Lie group with Lie algebra $\mathfrak{g}$. *(6 points)*

Solution. Consider the Lie group $G = SL_{20}(\mathbb{C}) \times SO_{25}(\mathbb{C})$ of block matrices

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $A \in SL_{20}(\mathbb{C})$ and $B \in SO_{25}(\mathbb{C})$. The tangent space of G at the identity $\mathrm{Id}_{45\times 45}$ can be computed blockwise. More precisely, a curve through the identity in G has the form

$$\begin{bmatrix} \mathrm{Id}_{20\times 20} + tx & 0 \\ 0 & \mathrm{Id}_{25\times 25} + ty \end{bmatrix} \in G \tag{4}$$

for a small real parameter t. Taking the derivative at $t = 0$ of the matrix (4) respects the block structure. Thus $\mathrm{Lie}(G) \cong \mathfrak{sl}_{20} \oplus \mathfrak{o}_{25}$.

Remark. The solution takes full marks even if just the answer "$G = SL_{20}(\mathbb{C}) \times SO_{25}(\mathbb{C})$" is stated.

(b) Give an example of two non-isomorphic Lie groups (either real or complex) with isomorphic Lie algebras. Justify your assertions. *(6 points)*

Solution. Both real Lie groups $\mathbb{R}$ and S^1 have Lie algebra $\mathbb{R}$. However, $\mathbb{R}$ is not compact and S^1 is, so they cannot be homeomorphic.

Remark. It is also possible to consider a non-connected Lie group G and a connected component of G.

PROBLEM 4

Let G be the Lie group of invertible functions $\mathbb{R} \xrightarrow{f_{a,b}} \mathbb{R},\ f_{a,b}(x) = ax + b$. In other words

$$G = \mathbb{R}^* \times \mathbb{R}, \qquad f_{a,b} \rightsquigarrow (a, b)$$

with the group operation induced by composition of functions. Write out this operation explicitly by filling in the blanks below *(7 points)*

$$(a, b) \cdot (a', b') = \underline{(aa', ab' + b)}.$$

Determine $\mathfrak{g} = \mathrm{Lie}(G)$ and its Lie bracket. *(5 points)*

(You may express elements of G near the identity $1 \in G$ as $g = 1 + \varepsilon y$ where $y \in \mathfrak{g}$ and ε is infinitesimally small. Then by considering $gg'g^{-1} \in G$ for group elements $g = 1 + \varepsilon y$ and $g' = 1 + \varepsilon' y'$, the order $\varepsilon\varepsilon'$ term recovers the Lie bracket $[y, y'] \in \mathfrak{g}$)

Solution 1. The real Lie group G has dimension 2, so its Lie algebra is isomorphic to $\mathbb{R}^2$ as a vector space. The Lie algebra structure of $\mathrm{Lie}(G)$ is determined by the bracket of two generators $x = (1, 0)$ and $y = (0, 1)$. The product structure previously computed tells us that $(1, 0) \in G$ is the identity, and $(a, b)^{-1} = \left(\frac{1}{a}, \frac{-b}{a}\right)$. Then, let $\epsilon, \epsilon' > 0$ and consider two curves

$$\gamma_x(\epsilon) = (1, 0) + \epsilon x + O(\epsilon) = (1 + \epsilon, 0) + O(\epsilon)\ ,\ \gamma_y(\epsilon') = (1, 0) + \epsilon' y + O(\epsilon') = (1, \epsilon') + O(\epsilon')$$

in G. Then from Exercise Sheet 1 we can compute

$$[x, y] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{d}{d\epsilon'}\Big|_{\epsilon'=0} \gamma_x(\epsilon)\gamma_y(\epsilon')\gamma_x(\epsilon)^{-1} =$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{d}{d\epsilon'}\Big|_{\epsilon'=0} (1 + \epsilon, 0)(1, \epsilon')\left(\frac{1}{1+\epsilon}, 0\right) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{d}{d\epsilon'}\Big|_{\epsilon'=0} (1, \epsilon'(1 + \epsilon)) = (0, 1) = y\ .$$

Remark 1. We can consider either the commutator $ghg^{-1}h^{-1}$ or ghg^{-1} in the solution.

Remark 2. Observe that $\gamma_x(\epsilon)^{-1} = \gamma_x(-\epsilon) + O(\epsilon)$. Thus it is also possible to set up the above computation as

$$[x, y] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \frac{d}{d\epsilon'}\Big|_{\epsilon'=0} \gamma_x(\epsilon)\gamma_y(\epsilon')\gamma_x(-\epsilon)$$

and this gives the same result.

Solution 2. We observe that G is isomorphic to the subgroup of GL_2 consisting of 2×2 matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

It follows that we can describe the Lie algebra of G as a subalgebra of $\mathfrak{gl}_2$. In particular $x \in \mathrm{Lie}(G)$ is determined by the condition that $\mathrm{Id}_{2\times 2} + \epsilon x \in G$ for ϵ infinitesimally small. It follows that $\mathrm{Lie}(G)$ is made out of matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

as a and b vary in $\mathbb{R}$.

PROBLEM 5

(a) What do we mean when we say that a (complex, finite-dimensional) representation V of a (complex, finite-dimensional) semisimple Lie algebra is completely reducible?

Solution 1. A (complex, finite-dimensional) representation V of a (complex, finite-dimensional) semisimple Lie algebra is completely reducible if there is an isomorphism of representations

$$V \cong V_1 \oplus \cdots \oplus V_k,$$

where each $V_i, i = 1, \ldots, k$ is an irreducible representation, *i.e.* it has no nontrivial subrepresentation.

Solution 2. A (complex, finite-dimensional) representation V of a (complex, finite-dimensional) semisimple Lie algebra is completely reducible if any subrepresentation $W \subset V$ has a complement W' in V (*i.e.* a vector subspace $W' \subset V$ such that $V = W \oplus W'$) which is also a subrepresentation.

(we encountered several, essentially equivalent, formulations of complete reducibility; stating any one of them in the space above would be acceptable) *(7 points)*

(b) A (complex, finite-dimensional) Lie algebra $\mathfrak{g}$ is called reductive if $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple. Prove that the adjoint representation of a reductive Lie algebra $\mathfrak{g}$ is completely reducible. *(5 points)*

Solution 1. The kernel of the adjoint representation

$$\mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{g}) \tag{5}$$

is exactly the center $\mathfrak{z}(\mathfrak{g})$. This means that (5) descends to a faithful representation

$$\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \hookrightarrow \operatorname{End}(\mathfrak{g}). \tag{6}$$

In particular, the map (5) has the same image as (6). We deduce that a $\mathfrak{g}$ module is irreducible if and only if it is irreducible for $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. We know from point (a) that every representation of a (complex, finite-dimensional) semisimple Lie algebra is completely reducible, so we are done.

Solution 2. We use the isomorphism of vector spaces[1] $\mathfrak{g} \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{g})$ to write any element $x \in \mathfrak{g}$ uniquely as a sum

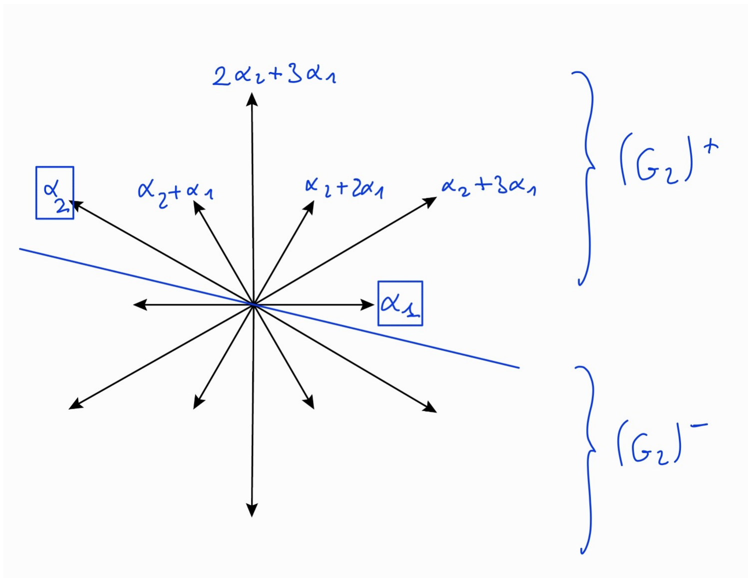
$$x = \overline{x} + z \tag{7}$$

[1]Actually Levi's theorem implies that it is an isomorphism of Lie algebras.

where $\overline{x}$ is the image of x in $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, and z is in the center. Let $W \subset \mathfrak{g}$ be a $\mathfrak{g}$-invariant subspace with respect to the adjoint action. Then W is also invariant with respect to the induced action $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \curvearrowright \mathfrak{g}$. Then we can find a complement $W' \subset \mathfrak{g}$ for W which is also invariant for the action of the semisimple Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$. Finally, decomposition (7) allows us to conclude that W is also a $\mathfrak{g}$-subrepresentation.

PROBLEM 6

The following is a picture of the root system of type G_2 (the long vectors are $\sqrt{3}$ times bigger than the short vectors, and the angles between adjacent vectors are all equal).



(a) Draw any half-plane whose boundary line passes through the origin but does not contain any of the root vectors: the 6 positive roots are those ones which lie in the chosen half plane.

Indicate in the picture the 2 simple roots α_1 and α_2 corresponding to your choice of half-plane.

Write on the picture the other 4 positive roots as explicit linear combinations of α_1 and α_2.

(9 points)

(b) Determine, with proof, the Weyl group corresponding to the root system of type G_2.

(6 points)

Solution 1. The Weyl group W of G_2 is generated by the two simple reflections $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$, for any possible choice of simple roots α_1, α_2. The generator s_i is the reflection with respect to the orthogonal line to the vector α_i, so in particular it is an isometry. We deduce that W acts on roots of the same length. Moreover, we can see that the action of W on the hexagon determined by the shortest (resp. longest) roots is transitive. Thus we conclude that W is isomorphic to the dihedral group D_6.

Solution 2. We work out the Weyl group W by generators and relations[2]. Possibly after rescaling, we may identify α_1 with $(1,0) \in \mathbb{R}^2$, and we write the reflections $s_1 = s_{\alpha_1}$ and $s_2 = s_{\alpha_2}$ with respect to the basis $((1,0),(0,1))$ of $\mathbb{R}^2$. We get

$$s_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad s_2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Then we get $s_1^2 = \mathrm{Id}_{2\times 2} = s_2^2$ and the order of $s_1 s_2$ is 6. Moreover $s_1(s_1 s_2)s_1 = (s_1 s_2)^{-1}$. Thus we get a homomorphism

$$\langle r, s\rangle/(s^2 = 1, r^6 = 2, srs = r^{-1}) = D_6 \to W$$

by sending $r \mapsto s_1 s_2$, $s \mapsto s_1$. Finally, since the Weyl group W acts transitively and faithfully on Weyl chambers, we see that it has 12 elements, just as D_6. This implies that $W \cong D_6$.

Remark. One could work out the above relations even without writing s_1 and s_2 in matrix form. For example, one could consider the action on Weyl chambers.

[2]Of course the explicit expressions in this solution depend on the choices made in point (a), but they lead to the same result.

PROBLEM 7

For any semisimple (complex, finite-dimensional) Lie algebra $\mathfrak{g}$ with root decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

consider the subspace

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

defined with respect to some decomposition $R = R^+ \sqcup R^-$ into positive and negative roots.

(a) Show that $\mathfrak{n}^+$ is a Lie subalgebra of $\mathfrak{g}$. *(9 points)*

Solution. The subset $\mathfrak{n}^+$ is clearly a linear subspace. Moreover, take $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ for any positive roots $\alpha, \beta \in R^+$. Then

$$[x, y] \in \mathfrak{g}_{\alpha+\beta} \ , \tag{8}$$

where $\alpha + \beta$ is again a positive root. Then the fact that the Lie bracket is linear concludes.

(b) Show that $\mathfrak{n}^+$ is a nilpotent Lie algebra. *(6 points)*

Solution 1. We use Engel's theorem. Take $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ for any positive roots $\alpha, \beta \in R^+$. Then applying equation (8) n times gives

$$\mathrm{ad}_x^n(y) \in \mathfrak{g}_{\beta+n\alpha} \quad .$$

Since the Lie algebra $\mathfrak{g}$ is finite dimensional, the above root space eventually vanishes, so that $\mathrm{ad}_x^n(y) = 0$ for n big enough.

Solution 2. We have to check that

$$\overbrace{[\mathfrak{n}^+, [\mathfrak{n}^+, [\dots, [\mathfrak{n}^+, \mathfrak{n}^+] \dots]]]}^{\text{n times}} = 0$$

for large n enough. Let $x_1 \in \mathfrak{g}_{\alpha_1}, \dots, x_n \in \mathfrak{g}_{\alpha_n}$ for positive roots $\alpha_1, \dots, \alpha_n \in R^+$. Then equation (8) implies that the bracket $[x_1, [x_2, \dots, [x_{n-1}, x_n] \dots]$ lies in the root space $\mathfrak{g}_{\alpha_1+\dots+\alpha_n}$. On the other hand, if we choose n big enough, any sum

$$\alpha_1 + \dots + \alpha_n \tag{9}$$

of n positive roots does not belong to the root system R. In fact, let N^+ be the number of positive roots. If we choose $n = kN^+$, then at least one root occurs at least k times in the sum (9). Let the aforementioned root be α_1. Thus the sum (9) may be written as

$$\beta + k\alpha_1$$

where $\beta \in R^+$. Since the Lie algebra $\mathfrak{g}$ is finite dimensional, the root space $\mathfrak{g}_{\beta+k\alpha_1}$ vanishes for k big enough, and for each pair β, α_1 of positive roots.

Remark. It is also possible to complete Solution 2 by observing that the coefficients of the decomposition into a sum of simple roots of

$$\alpha_1 + \cdots + \alpha_j$$

for $\alpha_1, \ldots, \alpha_j \in R^+$ increase with j.